

The dark side of gravity: Modified theories of gravity

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Modern astrophysical and cosmological models are faced with two severe theoretical difficulties, that can be summarized as the dark energy and the dark matter problems. Relative to the former, it has been stated that cosmology has entered a ‘golden age’, in which high-precision observational data have confirmed with startling evidence that the Universe is undergoing a phase of accelerated expansion. Several candidates, responsible for this expansion, have been proposed in the literature, in particular, dark energy models and modified gravity, amongst others. One is liable to ask: What is the so-called ‘dark energy’ that is driving the acceleration of the universe? Is it a vacuum energy or a dynamical field (“quintessence”)? Or is the acceleration due to infra-red modifications of Einstein’s theory of General Relativity? In the context of dark matter, two observations, namely, the behavior of the galactic rotation curves and the mass discrepancy in galactic clusters, suggest the existence of a (non or weakly interacting) form of dark matter at galactic and extra-galactic scales. It has also been proposed that modified gravity can explain the galactic dynamics without the need of introducing dark matter. We briefly review some of the modified theories of gravity that address these two intriguing and exciting problems facing modern physics.

I. INTRODUCTION

Cosmology is said to be thriving in a golden age, where a central theme is the perplexing fact that the Universe is undergoing an accelerating expansion [1]. The latter, one of the most important and challenging current problems in cosmology, represents a new imbalance in the governing gravitational equations. Historically, physics has addressed such imbalances by either identifying sources that were previously unaccounted for, or by altering the governing equations. The cause of this acceleration still remains an open and tantalizing question.

The standard model of cosmology has favored the first route to addressing the imbalance, namely, a missing stress-energy component. In particular, the dark energy models are fundamental candidates responsible for the cosmic expansion (see Refs. [2] for a review and references therein). A simple way to parameterize the dark energy is by an equation of state of the form $\omega \equiv p/\rho$, where p is the spatially homogeneous pressure and ρ the energy density of the dark energy. A value of $\omega < -1/3$ is required for cosmic expansion, as dictated by the Friedmann equation $\ddot{a}/a = -4\pi G(p + \rho/3)$, and $\omega = -1$ corresponds to a cosmological constant. A possibility that has been widely explored, is that of quintessence, a cosmic scalar field ϕ that has not yet reached the minimum of its potential $V(\phi)$ [3]. A common example is the energy of a slowly evolving scalar field with positive potential energy, similar to the inflaton field used to describe the inflationary phase of the Universe. In quintessence models the parameter range is $-1 < \omega < -1/3$, and the dark energy decreases with a scale factor $a(t)$ as $\rho_Q \propto a^{-3(1+\omega)}$ [4]. A specific exotic form of dark energy denoted phantom energy, with $\omega < -1$, has also been proposed [5], and possesses peculiar properties, such as the violation of the energy conditions and an infinitely increasing energy density. However, recent fits to supernovae, cosmic microwave background radiation (CMBR) and weak gravitational lensing data indicate that an evolving equation of state crossing the phantom divide, is mildly favored, and several models have been proposed in the literature [6, 7]. In particular, models considering a redshift dependent equation of state, possibly provide better fits to the most recent and reliable SN Ia supernovae Gold dataset.

It is also interesting to test specific models that are motivated by particle physics against the SN data, rather than trying to fit the phenomenological equations of state. In a cosmological setting, it has also been shown that the transition into the phantom regime, for a single field is probably physically implausible [6], so that a mixture of various interacting non-ideal fluids is necessary. If confirmed in the future, this behavior has important implications for theoretical models of dark energy. For instance, this implies that dark energy is dynamical and excludes the cosmological constant and the models with a constant parameter as possible candidates for dark energy. An alternative model to dark energy is that of the generalized Chaplygin gas (GCG), based on a negative pressure fluid, which is

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inversely proportional to the energy density [8], i.e., $p_{\text{ch}} = -A/\rho_{\text{ch}}^\alpha$, where A and α are positive constants. An attractive feature of this model, is that at early times, the energy density behaves as matter, $\rho_{\text{ch}} \sim a^{-3}$, where a is the scale factor, and as a cosmological constant at a later stage, $\rho_{\text{ch}} = \text{const}$. This dual behavior is responsible for the interpretation that the GCG model is a candidate of a unified model of dark matter and dark energy [9], and probably contains some of the key ingredients in the dynamics of the Universe for early and late times. All of these models present an extremely fascinating aspect for future experiments focussing on supernovae, CMBR and weak gravitational lensing and for future theoretical research.

One may also explore the alternative viewpoint, namely, through a modified gravity approach. A very promising way to explain these major problems is to assume that at large scales Einstein's theory of General Relativity breaks down, and a more general action describes the gravitational field. The Einstein field equation of General Relativity was first derived from an action principle by Hilbert, by adopting a linear function of the scalar curvature, R , in the gravitational Lagrangian density. However, there are no a priori reasons to restrict the gravitational Lagrangian to this form, and indeed several generalizations of the Einstein-Hilbert Lagrangian have been proposed, including "quadratic Lagrangians", involving second order curvature invariants such as R^2 , $R_{\mu\nu}R^{\mu\nu}$, $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$, $\varepsilon^{\alpha\beta\mu\nu}R_{\alpha\beta\gamma\delta}R_{\mu\nu}^{\gamma\delta}$, $C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}$, etc [10]. The physical motivations for these modifications of gravity were related to the possibility of a more realistic representation of the gravitational fields near curvature singularities and to create some first order approximation for the quantum theory of gravitational fields. In this context, a more general modification of the Einstein-Hilbert gravitational Lagrangian density involving an arbitrary function of the scalar invariant, $f(R)$, was considered in [11], and further developed in [12, 13, 14]. Recently, a renaissance of $f(R)$ modified theories of gravity has been verified in an attempt to explain the late-time accelerated expansion of the Universe. Earlier interest in $f(R)$ theories was motivated by inflationary scenarios as for instance, in the Starobinsky model, where $f(R) = R - \Lambda + \alpha R^2$ was considered [15]. In particular, it was shown that cosmic acceleration can be indeed explained with the context of $f(R)$ gravity [16], and the conditions of viable cosmological models have also been derived [17]. In the context of the Solar System regime, severe weak field constraints seem to rule out most of the models proposed so far [18, 19], although viable models do exist [20].

In the context of dark matter, the possibility that the galactic dynamics of massive test particles may be understood without the need for dark matter was also considered in the framework of $f(R)$ gravity models [21, 22, 23, 24, 25], and connections with MOND and the pioneer anomaly further explored by considering an explicit coupling of an arbitrary function of R with the matter Lagrangian density [26, 27]. The issue of dark matter is a long outstanding problem in modern astrophysics. Two observational aspects, namely, the behavior of the galactic rotation curves and the mass discrepancy in clusters of galaxies led to the necessity of considering the existence of dark matter at a galactic and extra-galactic scale. The rotation curves of spiral galaxies show that the rotational velocities increase from the center of the galaxy and then attain an approximately constant value, $v_{tg\infty} \sim 200 - 300$ km/s, within a distance r from the center of the galaxy [28]. In these regions the mass increases linearly with the radius, even where very little luminous matter can be detected. Relatively to the mass discrepancy in clusters of galaxies, the total mass of a cluster can be estimated in two ways. First, by taking into account the motions of its member galaxies, the virial theorem provides an estimate, M_V . Second, the total baryonic mass M may be estimated by considering the total sum of each individual member's mass. The mass discrepancy arises as one generally verifies that M_V is considerably greater than M , with typical values of $M_V/M \sim 20 - 30$ [28]. This is usually explained by postulating the existence of a dark matter, assumed to be a cold pressure-less medium distributed in a spherical halo around the galaxies.

Still in the context of modified gravity, an interesting possibility is the existence of extra dimensions. It is widely believed that string theory is moving towards a viable quantum gravity theory, and one of the key predictions of string theory is precisely the existence of extra spatial dimensions. In the brane-world scenario, motivated by recent developments in string theory, the observed 3-dimensional universe is embedded in a higher-dimensional spacetime [29]. Most brane-world models, including those of the Randall-Sundrum type [30], produce ultra-violet modifications to General Relativity, with extra-dimensional gravity dominating at high energies. However it is also possible for extra-dimensional gravity to dominate at low energies, leading to infra-red modifications of General Relativity. New features emerge in the brane scenario that may be more successful in providing a covariant infra-red modification of General Relativity, where it is possible for extra-dimensional gravity to dominate at low energies.

The Dvali-Gabadadze-Porrati (DGP) models [31] achieve this via a brane induced gravity effect. The generalization of the DGP models to cosmology lead to late-accelerating cosmologies [32], even in the absence of a dark energy field [33]. This exciting feature of "self acceleration" may help towards a new resolution to the dark energy problem, although this model deserves further investigation as a viable cosmological model [34]. While the DGP braneworld offers an alternative explanation to the standard cosmological model, for the expansion history of the universe, it offers a paradigm for nature fundamentally distinct from dark energy models of cosmic acceleration, even those that perfectly mimic the same expansion history. It is also fundamental to understand how one may differentiate this modified theory of gravity from dark energy models. The DGP braneworld theory also alters the gravitational interaction itself, yielding unexpected phenomenological extensions beyond the expansion history. Tests from the

solar system, large scale structure, lensing all offer a window into understanding the perplexing nature of the cosmic acceleration and, perhaps, of gravity itself [34]. The structure formation [35] and the inclusion of inflation are also important requirements of DGP gravity, if it is to be a realistic alternative to the standard cosmological model. Generalizations of the DGP model with the inclusion of a Gauss-Bonnet (GB) term have also been explored [33], the global structure of the DGP cosmologies have been analyzed [36], and research into other brane-world approaches to dark energy, such as the supersymmetric large extra dimensions (SLED) model have been undertaken [37].

In this work, we review several modified theories of gravity, exploring some of their interesting properties and characteristics, in particular, focussing mainly on the late-time cosmic acceleration. We refer the reader to excellent reviews, for instance, on dark energy and modified gravity in Ref. [2], $f(R)$ gravity in Ref. [38], an introduction to modified gravity as an alternative for dark energy in Ref. [39], and the late-time acceleration in braneworlds in Ref. [40]. This paper is outlined in the following manner. In Section II, we review $f(R)$ modified theories of gravity, for instance, focussing on the scalar-tensor representation of $f(R)$ and on the late-time acceleration; we further analyze an interesting extension to $f(R)$ gravity by considering an R -matter coupling, which yields some intriguing properties; we also discuss Gauss-Bonnet gravity and the late-time acceleration, and briefly consider modified Gauss-Bonnet gravity; the DGP brane-world model is also briefly reviewed. In Section III, we consider the possibility of dark matter being a geometric effect in $f(R)$ gravity, focussing on the $f(R)$ generalized virial theorem, and its applications to galactic cluster observations. Finally in Section IV, we conclude with a summary and discussions.

II. MODIFIED THEORIES OF GRAVITY: LATE-TIME COSMIC ACCELERATION

A. $f(R)$ modified theories of gravity

A promising avenue that has been extensively investigated recently are the $f(R)$ modified theories of gravity, where the standard Einstein-Hilbert action is replaced by an arbitrary function of the Ricci scalar R [41]. In this work, we use the metric formalism, which consists in varying the action with respect to $g^{\mu\nu}$, although other alternative approaches have been considered in the literature, namely, the Palatini formalism [42, 43], where the metric and the connections are treated as separate variables; and the metric-affine formalism, where the matter part of the action now depends and is varied with respect to the connection [43].

1. Action and field equations

The action for the $f(R)$ modified theories of gravity is given by

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S_M(g^{\mu\nu}, \psi), \quad (1)$$

where $\kappa = 8\pi G$. $S_M(g^{\mu\nu}, \psi)$ is the matter action, defined as $S_M = \int d^4x \sqrt{-g} \mathcal{L}_m(g_{\mu\nu}, \psi)$, where \mathcal{L}_m is the matter Lagrangian density, in which matter is minimally coupled to the metric $g_{\mu\nu}$ and ψ collectively denotes the matter fields.

Using the metric approach, by varying the action with respect to $g^{\mu\nu}$, provides the following field equation

$$F R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - \nabla_\mu \nabla_\nu F + g_{\mu\nu} \square F = \kappa T_{\mu\nu}^{(m)}, \quad (2)$$

where $F \equiv df/dR$. The matter stress-energy tensor, $T_{\mu\nu}^{(m)}$, is defined as

$$T_{\mu\nu}^{(m)} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_m)}{\delta(g^{\mu\nu})}. \quad (3)$$

Now, considering the contraction of Eq. (2), provides the following relationship

$$F R - 2f + 3 \square F = \kappa T, \quad (4)$$

which shows that the Ricci scalar is a fully dynamical degree of freedom.

Note that the field equation, Eq. (2), may be written as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}^{\text{eff}}, \quad (5)$$

where the effective stress energy tensor is given by $T_{\mu\nu}^{\text{eff}} = T_{\mu\nu}^{(c)} + \tilde{T}_{\mu\nu}^{(m)}$. The components $\tilde{T}_{\mu\nu}^{(m)}$ and the curvature stress-energy tensor, $T_{\mu\nu}^{(c)}$, are defined as

$$\tilde{T}_{\mu\nu}^{(m)} = T_{\mu\nu}^{(m)} / F, \quad (6)$$

$$T_{\mu\nu}^{(c)} = \frac{1}{\kappa F} \left[\nabla_\mu \nabla_\nu F - \frac{1}{4} g_{\mu\nu} (R F + \square F + \kappa T) \right], \quad (7)$$

respectively. It is also interesting to consider the conservation law for the above curvature stress-energy tensor. Taking into account the Bianchi identities, $\nabla^\mu G_{\mu\nu} = 0$, and the diffeomorphism invariance of the matter part of the action, which yields $\nabla^\mu T_{\mu\nu}^{(m)} = 0$, we verify that the effective Einstein field equation provides the following conservation law

$$\nabla^\mu T_{\mu\nu}^{(c)} = \frac{1}{F^2} T_{\mu\nu}^{(m)} \nabla^\mu F. \quad (8)$$

2. Scalar-tensor representation for $f(R)$ gravity

$f(R)$ gravity may be written as a scalar-tensor theory, by introducing a Legendre transformation $\{R, f\} \rightarrow \{\phi, V\}$ defined as

$$\phi \equiv F(R), \quad V(\phi) \equiv R(\phi) F - f(R(\phi)). \quad (9)$$

In this representation the field equations of $f(R)$ gravity can be derived from a Brans-Dicke type action with parameter $\omega = 0$, given by

$$S = \frac{1}{2\kappa} \int [\phi R - V(\phi) + L_m] \sqrt{-g} d^4x. \quad (10)$$

The only requirement for the gravitational field equations to be expressed in the form of a Brans-Dicke theory is that $F(R)$ be invertible, that is, $R(F)$ exists [19].

Thus, the field equations of $f(R)$ gravity can be reformulated as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi \frac{G}{\phi} T_{\mu\nu} + \theta_{\mu\nu}, \quad (11)$$

where

$$\theta_{\mu\nu} = -\frac{1}{2} V(\phi) g_{\mu\nu} + \frac{1}{\phi} (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) \phi. \quad (12)$$

Using these variables the trace equation, i.e., Eq. (4), takes the form

$$3\square\phi + 2V(\phi) - \phi \frac{dV}{d\phi} = \kappa T. \quad (13)$$

The modification of the standard Einstein-Hilbert action leads to the appearance of an effective gravitational constant $G_{\text{eff}} = G/\phi$ in the field equations. Note the presence of a new effective source term for the gravitational field, given by the tensor $\theta_{\mu\nu}$.

3. R -matter couplings in $f(R)$ gravity

Recently, in the context of $f(R)$ theories of modified gravity, it was shown that a function of R -matter coupling induces a non-vanishing covariant derivative of the stress-energy, $\nabla_\mu T^{\mu\nu} \neq 0$. This potentially leads to a deviation from geodesic motion, and consequently the appearance of an extra force [26]. Implications, for instance, for stellar equilibrium have been studied in Ref. [27]. The equivalence with scalar-tensor theories with two scalar fields has been considered in Ref. [44], and a viability stability criterion was also analyzed in Ref. [45]. It is interesting to note that nonlinear couplings of matter with gravity were analyzed in the context of the accelerated expansion of the Universe [46], and in the study of the cosmological constant problem [47].

The action for R -matter couplings, in $f(R)$ modified theories of gravity [26], takes the following form

$$S = \int \left\{ \frac{1}{2} f_1(R) + [1 + \lambda f_2(R)] \mathcal{L}_m \right\} \sqrt{-g} d^4x, \quad (14)$$

where $f_i(R)$ (with $i = 1, 2$) are arbitrary functions of the curvature scalar R . For notational simplicity we consider $\kappa = 1$ throughout this subsection.

Varying the action with respect to the metric $g^{\mu\nu}$ yields the field equations, given by

$$F_1 R_{\mu\nu} - \frac{1}{2} f_1 g_{\mu\nu} - \nabla_\mu \nabla_\nu F_1 + g_{\mu\nu} \square F_1 = -2\lambda F_2 \mathcal{L}_m R_{\mu\nu} + 2\lambda (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) \mathcal{L}_m F_2 + (1 + \lambda f_2) T_{\mu\nu}^{(m)}, \quad (15)$$

where we have denoted $F_i(R) = f'_i(R)$, and the prime represents the derivative with respect to the scalar curvature.

Now, taking into account the generalized Bianchi identities [26, 48], one deduces the following corrected conservation equation

$$\nabla^\mu T_{\mu\nu}^{(m)} = \frac{\lambda F_2}{1 + \lambda f_2} \left[g_{\mu\nu} \mathcal{L}_m - T_{\mu\nu}^{(m)} \right] \nabla^\mu R, \quad (16)$$

where the coupling between the matter and the higher derivative curvature terms describes an exchange of energy and momentum between both.

In the following, consider the equation of state for a perfect fluid

$$T_{\mu\nu}^{(m)} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}, \quad (17)$$

where ρ is the energy density and p , the pressure, respectively. The four-velocity, U_μ , satisfies the conditions $U_\mu U^\mu = -1$ and $U^\mu U_{\mu;\nu} = 0$.

Introducing the projection operator $h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$, gives rise to non-geodesic motion governed by the following equation of motion for a fluid element: $dU^\mu/ds + \Gamma_{\alpha\beta}^\mu U^\alpha U^\beta = f^\mu$, where the extra force, f^μ , is given by

$$f^\mu = \frac{1}{\rho + p} \left[\frac{\lambda F_2}{1 + \lambda f_2} (\mathcal{L}_m - p) \nabla_\nu R + \nabla_\nu p \right] h^{\mu\nu}. \quad (18)$$

In a recent paper [49], the authors argued that a “natural choice” for the matter Lagrangian density for perfect fluids is $\mathcal{L}_m = p$, based on Refs. [50, 51], where p is the pressure. This choice has a particularly interesting application in the analysis of the R -matter coupling for perfect fluids, which implies in the vanishing of the extra force [26]. However, it is important to point out that despite the fact that $\mathcal{L}_m = p$ does indeed reproduce the perfect fluid equation of state, it is not unique [52]. Other choices include, for instance, $\mathcal{L}_m = -\rho$ [51, 53], where ρ is the energy density, or $\mathcal{L}_m = -na$, where n is the particle number density, and a is the physical free energy defined as $a = \rho/n - Ts$, with T being the temperature and s the entropy per particle (see Ref. [51, 52] for details).

Hence, it is clear that no immediate conclusion may be extracted regarding the additional force imposed by the non-minimum coupling of curvature to matter, given the different available choices for the Lagrangian density. One may conjecture that there is a deeper principle or symmetry that provides a unique Lagrangian density for a perfect fluid [52]. This has not been given due attention in the literature, as arbitrary gravitational field equations depending on the matter Lagrangian have not always been the object of close analysis. See Ref. [52] for more details.

4. Late-time cosmic acceleration

In this subsection, we show that $f(R)$ gravity may lead to an effective dark energy, without the need to introduce a negative pressure ideal fluid. Consider the FLRW metric

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right]. \quad (19)$$

Taking into account the perfect fluid description for matter given by Eq. (17), we verify that the gravitational field equation, Eq. (5), provides the generalised Friedmann equations in the following form [54, 55]:

$$\left(\frac{\dot{a}}{a} \right)^2 - \frac{1}{3F(R)} \left\{ \frac{1}{2} [f(R) - RF(R)] - 3 \left(\frac{\dot{a}}{a} \right) \dot{R} F'(R) \right\} = \frac{\kappa}{3} \rho, \quad (20)$$

$$\left(\frac{\ddot{a}}{a} \right) + \frac{1}{2F(R)} \left\{ \dot{a} \dot{R} F'(R) + \ddot{R} F'(R) + \dot{R}^2 F''(R) - \frac{1}{3} [f(R) - RF(R)] \right\} = -\frac{\kappa}{6} (\rho + 3p). \quad (21)$$

These modified Friedmann field equations may be rewritten in a more familiar form, as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3}\rho_{\text{tot}}, \quad (22)$$

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{\kappa}{6}(\rho_{\text{tot}} + 3p_{\text{tot}}), \quad (23)$$

where $\rho_{\text{tot}} = \rho + \rho_{(c)}$ and $p_{\text{tot}} = p + p_{(c)}$, and the curvature stress-energy components, $\rho_{(c)}$ and $p_{(c)}$, are defined as

$$\rho_{(c)} = \frac{1}{\kappa F(R)} \left\{ \frac{1}{2} [f(R) - RF(R)] - 3 \left(\frac{\dot{a}}{a} \right) \dot{R}F'(R) \right\}, \quad (24)$$

$$p_{(c)} = \frac{1}{\kappa F(R)} \left\{ 2 \left(\frac{\dot{a}}{a} \right) \dot{R}F'(R) + \ddot{R}F'(R) + \dot{R}^2 F''(R) - \frac{1}{2} [f(R) - RF(R)] \right\}, \quad (25)$$

respectively. The late-time cosmic acceleration is achieved if the condition $\rho_{\text{tot}} + 3p_{\text{tot}} < 0$ is obeyed, which follows from Eq. (23).

For simplicity, consider the absence of matter, $\rho = p = 0$. Now, taking into account the equation of state $\omega_{\text{eff}} = p_{(c)}/\rho_{(c)}$, with $f(R) \propto R^n$ and a generic power law $a(t) = a_0(t/t_0)^\alpha$ [54], the parameters ω_{eff} and α are given by

$$\omega_{\text{eff}} = -\frac{6n^2 - 7n - 1}{6n^2 - 9n + 3}, \quad \alpha = \frac{-2n^2 + 3n - 1}{n - 2}, \quad (26)$$

respectively, for $n \neq 1$. Note that a suitable choice of n can lead to the desired value of $\omega_{\text{eff}} < -1/3$, achieving the late-time cosmic acceleration.

Another example is a model of the form $f(R) = R - \mu^{2(n+1)}/R^n$ [16]. Choosing once again a generic power law for the scale factor, the parameter can be written as

$$\omega_{\text{eff}} = -1 + \frac{2(n+2)}{3(2n+1)(n+1)}, \quad (27)$$

and once again a desired value of $\omega_{\text{eff}} < -1/3$ may be attained, by appropriately choosing the value of the parameter n . Note that as $n \rightarrow \infty$ the spacetime is approximately de Sitter.

Other forms of $f(R)$ have also been considered in the literature, for instance those involving logarithmic terms, such as $f(R) = R + \alpha \ln(R/\mu^2) + \beta R^m$ or $f(R) = R + \gamma R^{-n} [\ln(R/\mu^2)]^m$ [39, 56]. These models also yield acceptable values for the effective equation of state parameter, resulting in the late-time cosmic acceleration.

B. Gauss-Bonnet gravity and cosmic acceleration

1. Gauss-Bonnet gravity

In considering alternative higher-order gravity theories, one is liable to be motivated in pursuing models consistent and inspired by several candidates of a fundamental theory of quantum gravity. In this context, it may be possible that unusual gravity-matter couplings predicted by string/M-theory may become important at the recent low-curvature Universe. For instance, one may couple a scalar field not only with the curvature scalar, as in scalar-tensor theories, but also with higher order curvature invariants. Indeed, motivations from string/M-theory predict that scalar field couplings with the Gauss-Bonnet invariant \mathcal{G} are important in the appearance of non-singular early time cosmologies. It is also possible to apply these motivations to the late-time Universe in an effective Gauss-Bonnet dark energy model [57].

Consider the action of Gauss-Bonnet gravity given by

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa} - \frac{\lambda}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) + f(\phi) \mathcal{G} \right] + S_M(g^{\mu\nu}, \psi), \quad (28)$$

where $\lambda = +1$ is defined for a canonical scalar field, and $\lambda = -1$ for a phantom field, respectively. The Gauss-Bonnet invariant is given by $\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. It is also important to note that in the matter action, matter is minimally coupled to the metric and not to the scalar field, making Gauss-Bonnet gravity a metric theory. Thus, using the diffeomorphism invariance of $S_M(g^{\mu\nu}, \psi)$ yields the covariant conservation of the stress-energy tensor, $\nabla^\mu T_{\mu\nu}^{(m)} = 0$.

Varying the action with respect to ϕ , provides the equation of motion for the scalar field, given by

$$\lambda \nabla^2 \phi - V'(\phi) + f'(\phi) \mathcal{G} = 0. \quad (29)$$

The gravitational field equations are given by varying the action with respect to the metric $g^{\mu\nu}$, and provides the following relationship [55, 57]

$$\begin{aligned} & \frac{1}{\kappa} G^{\mu\nu} - \frac{1}{2} g^{\mu\nu} f(\phi) \mathcal{G} + 2f(\phi) R R^{\mu\nu} - 2\nabla^\mu \nabla^\nu [f(\phi) R] + 2g^{\mu\nu} \nabla^2 [f(\phi) R] \\ & - 8f(\phi) R^\mu{}_\rho R^{\nu\rho} + 4\nabla_\rho \nabla^\mu [f(\phi) R^{\nu\rho}] + 4\nabla_\rho \nabla^\nu [f(\phi) R^{\mu\rho}] - 4\nabla^2 [f(\phi) R^{\mu\nu}] \\ & - 4g^{\mu\nu} \nabla_\rho \nabla_\sigma [f(\phi) R^{\rho\sigma}] + 2f(\phi) R^{\mu\rho\sigma\tau} R^\nu{}_{\rho\sigma\tau} - 4\nabla_\rho \nabla_\sigma [f(\phi) R^{\mu\rho\sigma\nu}] = T^{\mu\nu} + T_\phi^{\mu\nu}, \end{aligned} \quad (30)$$

where $T_\phi^{\mu\nu}$ is given by

$$T_\phi^{\mu\nu} = \lambda \left(\frac{1}{2} \partial^\mu \phi \partial^\nu \phi - \frac{1}{4} g^{\mu\nu} \partial_\rho \phi \partial^\rho \phi \right) - \frac{1}{2} g^{\mu\nu} V(\phi). \quad (31)$$

One may use the following identities obtained from the Bianchi identity [55, 57]:

$$\begin{aligned} \nabla^\rho R_{\rho\tau\mu\nu} &= \nabla_\mu R_{\nu\tau} - \nabla_\nu R_{\mu\tau}, \quad \nabla^\nu R_{\nu\mu} = \frac{1}{2} \nabla_\mu R, \quad \nabla_\mu \nabla_\nu R^{\mu\nu} = \frac{1}{2} \square R, \\ \nabla_\rho \nabla_\sigma R^{\mu\rho\nu\sigma} &= \nabla^2 R^{\mu\nu} - \frac{1}{2} \nabla^\mu \nabla^\nu R + R^{\mu\rho\nu\sigma} R_{\rho\sigma} - R^\mu{}_\rho R^{\nu\rho}, \\ \nabla_\rho \nabla^{(\mu} R^{\nu)\rho} &= \frac{1}{2} \nabla^{(\mu} \nabla^{\nu)} R - R^{\mu\rho\nu\sigma} R_{\rho\sigma} + R^\mu{}_\rho R^{\nu\rho}, \end{aligned}$$

in the Gauss-Bonnet gravitational field equation, which may then be formally simplified to

$$\begin{aligned} & \frac{1}{\kappa} G^{\mu\nu} - \frac{1}{2} g^{\mu\nu} f(\phi) \mathcal{G} + 2f(\phi) R R^{\mu\nu} + 4f(\phi) R^\mu{}_\rho R^{\nu\rho} \\ & + 2f(\phi) R^{\mu\rho\sigma\tau} R^\nu{}_{\rho\sigma\tau} - 4f(\phi) R^{\mu\rho\sigma\nu} R_{\rho\sigma} = T^{\mu\nu} + T_\phi^{\mu\nu} + T_c^{\mu\nu}, \end{aligned} \quad (32)$$

where $T_c^{\mu\nu}$ is defined as

$$\begin{aligned} T_c^{\mu\nu} &= 2[\nabla^\mu \nabla^\nu f(\phi)] R - 2g^{\mu\nu} [\nabla^2 f(\phi)] R - 4[\nabla_\rho \nabla^\mu f(\phi)] R^{\nu\rho} - 4[\nabla_\rho \nabla^\nu f(\phi)] R^{\mu\rho} \\ &+ 4[\nabla^2 f(\phi)] R^{\mu\nu} + 4g^{\mu\nu} [\nabla_\rho \nabla_\sigma f(\phi)] R^{\rho\sigma} - 4[\nabla_\rho \nabla_\sigma f(\phi)] R^{\mu\rho\sigma\nu}. \end{aligned} \quad (33)$$

Using the FLRW metric, the modified Friedmann equations for Gauss-Bonnet gravity reduce to the following relationships

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{1}{3} \kappa \left(\rho + \frac{\lambda}{2} \dot{\phi}^2 + V(\phi) - 24\dot{\phi} f'(\phi) H^3 \right), \quad (34)$$

$$\left(\frac{\ddot{a}}{a} \right) = -\frac{\kappa}{6} (\rho + 3p) - \frac{\kappa}{3} \left[\lambda \dot{\phi}^2 - V(\phi) + 12H^3 \dot{\phi} f'(\phi) + 12 \frac{\partial}{\partial t} (H^2 \dot{f}) \right], \quad (35)$$

and the equation of motion for the scalar field is given by

$$\lambda \left(\ddot{\phi} + 3H\dot{\phi} \right) + V'(\phi) - 24f'(\phi) H^2 \left(\dot{H} + H^2 \right) = 0, \quad (36)$$

where the expression for the Gauss-Bonnet invariant, $\mathcal{G} = 24H^2(\dot{H} + H^2)$, is used.

Note that in the absence of matter, the Gauss-Bonnet gravitational field equations may be written as

$$\rho_{\text{GB}} = \frac{3}{\kappa} H^2, \quad p_{\text{GB}} = -\frac{1}{\kappa} (3H^2 + 2\dot{H}), \quad (37)$$

where the Gauss-Bonnet curvature stress-energy tensor components are defined as

$$\rho_{\text{GB}} = \frac{\lambda}{2} \dot{\phi}^2 + V(\phi) - 24\dot{\phi} f'(\phi) H^3, \quad (38)$$

$$p_{\text{GB}} = \frac{\lambda}{2} \dot{\phi}^2 - V(\phi) + 8 \frac{\partial}{\partial t} (H^2 \dot{f}) + 16H^3 \dot{\phi} f'(\phi). \quad (39)$$

These relationships are particularly interesting as one may now define an effective equation of state given by

$$\omega_{\text{eff}} = \frac{p_{\text{GB}}}{\rho_{\text{GB}}} = -1 - \frac{2\dot{H}}{3H^2}. \quad (40)$$

Consider the choices of an exponential scalar-GB coupling and exponential scalar potential given by [57]

$$V(\phi) = V_0 e^{-2\phi/\phi_0}, \quad f(\phi) = f_0 e^{2\phi/\phi_0}. \quad (41)$$

The scale factor and the scalar field are respectively chosen in the following form

$$a(t) = \begin{cases} a_0 t^{h_0}, & \text{for } h_0 > 0 \text{ (quintessence),} \\ a_0 (t_s - t)^{h_0}, & \text{for } h_0 < 0 \text{ (phantom),} \end{cases} \quad (42)$$

and

$$\phi(t) = \begin{cases} \phi_0 \ln\left(\frac{t}{t_1}\right), & \text{for } h_0 > 0, \\ \phi_0 \ln\left(\frac{t_s - t}{t_1}\right), & \text{for } h_0 < 0, \end{cases} \quad (43)$$

where t_1 is an arbitrary constant [57].

Using the above choices, and considering the absence of matter, then taking into account Eqs. (34) and (36), and finally reorganizing the terms leads to the following equations

$$V_0 t_1^2 = -\frac{1}{\kappa(1+h_0)} \left[3h_0^2(1-h_0) + \frac{\lambda\phi_0^2\kappa(1-5h_0)}{2} \right], \quad (44)$$

$$\frac{48f_0 h_0^2}{t_1^2} = -\frac{6}{\kappa(1+h_0)} \left(h_0 - \frac{\lambda\phi_0^2\kappa}{2} \right). \quad (45)$$

The effective equation of state parameter, Eq. (40), takes the following form

$$\omega_{\text{eff}} = -1 - \frac{2}{3h_0}. \quad (46)$$

Note that if $h_0 < 0$ then $\omega_{\text{eff}} < -1$, reflecting an effective phantom regime; and if $h_0 > 0$ then $\omega_{\text{eff}} > -1$, which reflects an effective quintessence regime. However, it has been shown that the case of $h_0 < 0$ is always stable, while the case of a non-phantom $h_0 > 0$ cosmology is always unstable [57]. An interesting example is the case of $V_0 = 0$, which from Eq. (44) imposes the following condition [57]: $\phi_0^2 = -6h_0^2(1-h_0)/[\lambda\kappa(1-5h_0)]$. In order for ϕ_0 to be real, in the case of the a canonical scalar, $\lambda = 1$, one finds that $1/5 < h_0 < 1$; for a phantom field, $\lambda = -1$, then $h_0 < 1/5$ or $h_0 \geq 1$. To achieve an effective equation of state parameter value that mimics dark energy is to consider, for instance, $h_0 = -80/3$, which imposes the $\omega_{\text{eff}} = -1.025$.

We refer the reader to Ref. [57] for more examples and details. It is also interesting to note that observational constraints [58], such as CMBR, galaxy distribution, large scale structure and supernovae seem to favour the Gauss-Bonnet coupling.

2. Modified Gauss-Bonnet gravity and the late-time acceleration

An interesting alternative gravitational theory is modified Gauss-Bonnet gravity, which is given by the following action:

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa} + f(\mathcal{G}) \right] + S_M(g^{\mu\nu}, \psi). \quad (47)$$

This theory has been extensively analyzed in the literature [39, 59, 60], and rather than review all of its intricate details here, we note that it is a subset of Gauss-Bonnet gravity given by the action (28).

To see this, we follow closely the approach outlined in Ref. [55, 60]. By introducing two auxiliary scalar fields A and B , the gravitational part of the action (47), may be rewritten as

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa} + B(\mathcal{G} - A) + f(A) \right]. \quad (48)$$

Now varying with respect to B , one obtains $A = \mathcal{G}$, so that the action (48) is recovered. Varying with respect to A , one obtains $B = f'(A)$, and substituting in (48) leads to

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa} + f'(A)(\mathcal{G} - A) + f(A) \right]. \quad (49)$$

With the following definitions

$$\phi = A, \quad \text{and} \quad V(\phi) = Af'(A) - f(A), \quad (50)$$

one finally ends up with the following action

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa} - V(\phi) + f(\phi)\mathcal{G} \right] + S_M(g^{\mu\nu}, \psi). \quad (51)$$

which is simply the action for Gauss-Bonnet gravity given by Eq. (28) with the absence of the kinetic term. Thus, modified Gauss-Bonnet theory, given by (47) is dynamically equivalent to Gauss-Bonnet gravity with $\lambda = 0$ [55, 60]. We refer the reader to Refs. [39, 59] for more details.

C. DGP brane gravity and self-acceleration

One of the key predictions of string theory is the existence of extra spatial dimensions. In the brane-world scenario, motivated by recent developments in string theory, the observed 3-dimensional universe is embedded in a higher-dimensional spacetime [29]. One of the simplest covariant models is the Dvali-Gabadadze-Porrati (DGP) braneworld model, in which gravity leaks off the 4D Minkowski brane into the 5D bulk at large scales. The generalization of the DGP models to cosmology lead to late-accelerating cosmologies [32], even in the absence of a dark energy field [33]. While the DGP braneworld offers an alternative explanation to the standard cosmological model, for the expansion history of the universe, it offers a paradigm for nature fundamentally distinct from dark energy models of cosmic acceleration, even those that perfectly mimic the same expansion history.

The 5D action describing the DGP model is given by

$$S = \frac{1}{2\kappa_5} \int d^5x \sqrt{-g} {}^{(5)}R + \frac{1}{2\kappa_4} \int d^4x \sqrt{-\gamma} {}^{(4)}R - \int d^4x \sqrt{-\gamma} \mathcal{L}_m, \quad (52)$$

where $\kappa_5 = 8\pi G_5$ and $\kappa_4 = 8\pi G_4$. The first term in the action is the Einstein-Hilbert action in five dimensions for a five-dimensional bulk metric, g_{AB} , with a five-dimensional Ricci scalar ${}^{(5)}R$; the second term is the induced Einstein-Hilbert term on the brane, with a four-dimensional induced metric γ on the brane; and \mathcal{L}_m represents the matter Lagrangian density confined to the brane.

The transition from 4D to 5D behavior is governed by a cross-over scale, r_c , given by

$$r_c = \frac{\kappa_5}{2\kappa_4}. \quad (53)$$

Gravity manifests itself as a 4-dimensional theory for characteristic scales much smaller than r_c ; for large distances compared to r_c , one verifies a leakage of gravity into the bulk, consequently making the higher dimensional effects important. Thus, the leakage of gravity at late times initiates acceleration, due to the weakening of gravity on the brane.

For weak fields, the gravitational potential behaves as

$$\Phi \sim \begin{cases} r^{-1}, & \text{for } r < r_c, \\ r^{-2}, & \text{for } r > r_c. \end{cases} \quad (54)$$

Taking into account the FRW metric, and considering a flat geometry, the modified Friedmann equation is given by

$$H^2 - \frac{\epsilon}{r_c} H = \frac{8\pi G}{3} \rho, \quad (55)$$

where $\epsilon = \pm 1$, and the energy density satisfies the standard conservation equation, i.e., $\dot{\rho} + 3H(\rho + p) = 0$. For scales $H^{-1} \ll r_c$, the second term is negligible, and Eq. (55) reduces to the general relativistic Friedmann equation,

i.e., $H^2 = 8\pi G\rho/3$. The second term becomes significant for scales comparable to the cross-over scale, $H^{-1} \geq r_c$. Self-acceleration occurs for the branch $\epsilon = +1$, and the modified Friedmann equation shows that at late times in a CDM universe characterized by a scale factor $\rho \propto a^{-3}$, the universe approaches a de Sitter solution

$$H \rightarrow H_\infty = \frac{1}{r_c} \quad (56)$$

Thus, one may achieve late-time acceleration if the H_0 is of the order of H_∞ . Note that the late-time acceleration in the DGP model is not due to the presence of a negative pressure, but simply due to the weakening of gravity on the brane as a consequence of gravity leakage at late times.

Although the weak-field gravitational DGP behaves as $4D$ on scales smaller than r_c , linearized DGP gravity is not described by General Relativity [40, 61]. It is also interesting to note that despite the fact that the expansion history of the DGP model and General Relativity are the same, the structure formation in both are essentially different [62]. Combining these features provides the possibility of distinguishing the DGP model from dark energy models in General Relativity. We should also emphasize that while the DGP model is not ruled out by current observations, the Λ CDM model fits the data comfortably [63]. Another interesting aspect of this model is that the self-accelerating branch in the DGP model contains a ghost at the linearized level [61, 64]. The presence of the ghost implies a negative sign for the kinetic term, resulting in negative energy densities, consequently leading to the instability of the spacetime. However, in a recent paper it was claimed that a higher codimension generalization of the DGP scenario is free of ghost instabilities [65], and further work along these lines is currently underway. We refer the reader to Ref. [40, 61], and references therein, for more details on the DGP model.

III. DARK MATTER AS A GEOMETRIC EFFECT OF MODIFIED GRAVITY

The issue of dark matter is a long outstanding problem in modern astrophysics. Two observational aspects, namely, the behavior of the galactic rotation curves and the mass discrepancy in clusters of galaxies led to the necessity of considering the existence of dark matter at a galactic and extra-galactic scale. The galactic rotation curves of spiral galaxies [28] are probably the most striking evidences for the possible failure of Newtonian gravity and of General Relativity on galactic and intergalactic scales. In these galaxies, neutral hydrogen clouds are observed at large distances from the center, much beyond the extent of the luminous matter. As these clouds are moving in circular orbits with nearly constant tangential velocity v_{tg} , such orbits are maintained by the balance between the centrifugal acceleration v_{tg}^2/r and the gravitational attraction $GM(r)/r^2$ of the total mass $M(r)$ contained within the radius r . This yields an expression for the galactic mass profile of the form $M(r) = rv_{\text{tg}}^2/G$, with the mass increasing linearly with r , even at large distances, where very little luminous matter has been detected [28]. This peculiar behavior of the rotation curves is usually explained by postulating the existence of dark matter, assumed to be a cold and pressureless medium, distributed in a spherical halo around the galaxies. There are many possible candidates for dark matter, the most popular ones being the weakly interacting massive particles (WIMP) [66].

One cannot also *a priori* exclude the possibility that Einstein's (and Newton's) theory of gravity breaks down at galactic scales. In this context, several theoretical models, based on a modification of Newton's law or of General Relativity, have been proposed to explain the behavior of the galactic rotation curves [67]. A promising avenue that has been extensively investigated recently are the $f(R)$ modified theories of gravity. In this context, early work in explaining dark matter using $f(R)$ gravity using models of the form $f(R) \propto R^n$ found large values for n [21, 22, 23].

In these papers, a power law modified Newtonian potential of the form $\Phi(r) = -\frac{Gm}{2r}[1 + (r/r_c)^\beta]$ was considered, to describe the observed behavior of the galactic rotation curves, where m is the mass of the particle, r_c a constant and the coefficient β depends on the 'slope' parameter n in the modified action. Using this modified Newtonian potential, it was found that the best fit to 15 low luminosity rotation curves in R^n gravity is obtained for $n = 3.5$ [21] (somewhat lower values, in particular, $n = 2.2$, were obtained in [22, 23]). These results seem to suggest that a strong modification of standard general relativity is required to explain the observed behavior of the galactic rotation curves. Note that these large values of n are in gross violation with the Solar System tests.

However, recently it was found that only slight deviations from General Relativity are needed, i.e., $n = 1 + \epsilon$ with $\epsilon \ll 1$ [24, 25]. This discrepancy of values can be traced back to the correction term of the modified Newtonian potential. It was shown in Ref. [24] that the correct modified term to the Newtonian potential in the "dark matter" dominated region, where the rotation curves are strictly flat, must have a logarithmic dependence on the radial coordinate r , of the form $\Phi_N(r) = -\frac{Gm}{2r} + v_{\text{tg}}^2 \ln(r/r_0)$, where r_0 is an arbitrary constant of integration (we refer the reader to Ref. [24] for details). These differences in the Newtonian limit in the two models result in different values of the parameter n in the power-law modified action of the gravity.

In the following sections, we shall analyze the 'dark matter' problem by considering a generalized version of the virial theorem in the framework of $f(R)$ modified theories of gravity [25]. Recall that due to its generality and wide

range of applications, the virial theorem plays an important role in astrophysics. Assuming steady state, one of the important results which can be obtained with the use of the virial theorem is to deduce the mean density of astrophysical objects such as galaxies, clusters and super clusters, by observing the velocities of test particles rotating around them. Hence the virial theorem can be used to predict the total mass of the clusters of galaxies.

The generalized virial theorem, in the context of $f(R)$ gravity is obtained by using a method based on the collisionless Boltzmann equation [25]. The additional geometric terms present in the modified gravitational field equations provide an effective contribution to the gravitational energy, which at the galactic/extra-galactic level acts as an effective mass, playing the role of the ‘dark matter’. The total virial mass of the galactic clusters is mainly determined by the effective mass associated to the new geometrical term, the geometrical mass. It is important to note that the latter term may account for the well-known virial theorem mass discrepancy in clusters of galaxies.

A. The generalized virial theorem in $f(R)$ gravity

Consider an isolated and spherically symmetric cluster described by a static and spherically symmetric metric

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2). \quad (57)$$

The galaxies, treated as identical and collisionless point particles, are described by a distribution function f_B , which obeys the general relativistic Boltzmann equation.

In terms of the distribution function the stress-energy tensor can be written as $T_{\mu\nu} = \int f_B m u_\mu u_\nu du$, where m is the mass of the particle (galaxy) [68], u_μ is the four-velocity of the galaxy and $du = du_r du_\theta du_\varphi / u_t$ is the invariant volume element of the velocity space. Thus, the stress-energy tensor of the matter in a cluster of galaxies can be represented in terms of an effective density ρ_{eff} and of an effective anisotropic pressure, with radial $p_{\text{eff}}^{(r)}$ and tangential $p_{\text{eff}}^{(\perp)}$ components, given by

$$\rho_{\text{eff}} = \rho \langle u_t^2 \rangle, \quad p_{\text{eff}}^{(r)} = \rho \langle u_r^2 \rangle, \quad p_{\text{eff}}^{(\perp)} = \rho \langle u_\theta^2 \rangle = \rho \langle u_\varphi^2 \rangle, \quad (58)$$

where, at each point, $\langle u_r^2 \rangle$ is the average value of u_r^2 , etc, and ρ is the mass density [69].

In what follows, we use this form of the stress-energy tensor, and for convenience take into account the scalar-tensor representation of $f(R)$ gravity outlined in Section II A 2. As we are interested in astrophysical applications at the extra-galactic level, we may assume that the deviations from standard General Relativity (corresponding to the background value $\phi = 1$) are small. Therefore we may represent ϕ as $\phi = 1 + \epsilon g'(R)$, where ϵ is a small quantity, and $g'(R)$ describes the modifications of the geometry due to the presence of the tensor $\theta_{\mu\nu}$ [19], so that $1/\phi \simeq 1 - \epsilon g'(R)$. Now adding up the non-zero components of the gravitational field equation Eq. (11), and taking into account the above approximations (see Ref. [25] for details), one obtains the following relationship

$$e^{-\lambda} \left(\frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\nu'}{r} - \frac{\nu' \lambda'}{4} \right) \simeq 4\pi G \rho \langle u^2 \rangle + 4\pi G \rho_\phi, \quad (59)$$

where $\langle u^2 \rangle = \langle u_t^2 \rangle + \langle u_r^2 \rangle + \langle u_\theta^2 \rangle + \langle u_\varphi^2 \rangle$, and the useful quantity ρ_ϕ is defined as

$$\rho_\phi \simeq -\epsilon \rho \langle u^2 \rangle g'(R) + \frac{1}{4\pi G} \left[\frac{1}{\phi} V(\phi) + \frac{1}{\phi} (2\nabla_t \nabla^t + \square) \phi \right] \Big|_{\phi=1+\epsilon g'(R)}, \quad (60)$$

which may be interpreted as the geometric energy density.

It is convenient to introduce some approximations that apply to test particles in stable circular motion around galaxies, and to the galactic clusters. First of all, we assume that ν and λ are slowly varying (i.e. ν' and λ' small), so that in Eq. (59) the quadratic terms can be neglected. Secondly, we assume that the galaxies have non-relativistic velocities, so that $\langle u_1^2 \rangle \approx \langle u_2^2 \rangle \approx \langle u_3^2 \rangle \ll \langle u_0^2 \rangle \approx 1$. Thus, Eq. (59) becomes

$$\frac{1}{2r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \nu}{\partial r} \right) = 4\pi G \rho + 4\pi G \rho_\phi. \quad (61)$$

In order to derive the virial theorem for galaxy clusters, one uses the relativistic Boltzmann equation, which provides the following relationship (see Ref. [25] for details)

$$2K - \frac{1}{2} \int_0^R 4\pi r^3 \rho \frac{\partial \nu}{\partial r} dr = 0, \quad \text{with} \quad K = \int_0^R 2\pi \rho [\langle u_1^2 \rangle + \langle u_2^2 \rangle + \langle u_3^2 \rangle] r^2 dr. \quad (62)$$

K is the total kinetic energy of the galaxies, and the total mass of the system is given by $M = \int_0^R dM(r) = \int_0^R 4\pi\rho r^2 dr$. The main contribution to M is due to the baryonic mass of the intra-cluster gas and of the stars, but other particles, such as massive neutrinos, may also contribute significantly to M .

Now, multiplying Eq. (61) by r^2 and integrating from 0 to r we obtain

$$GM(r) = \frac{1}{2}r^2 \frac{\partial \nu}{\partial r} - GM_\phi(r), \quad \text{with} \quad M_\phi(r) = 4\pi \int_0^r \rho_\phi(r') r'^2 dr'. \quad (63)$$

The useful quantity M_ϕ is denoted as the *geometric mass* of the cluster. By multiplying Eq. (63) with $dM(r)$, followed by an integration one deduces the relationship

$$\Omega = \Omega_\phi - \frac{1}{2} \int_0^R 4\pi r^3 \rho \frac{\partial \nu}{\partial r} dr, \quad (64)$$

with the following definitions

$$\Omega = - \int_0^R \frac{GM(r)}{r} dM(r), \quad \text{and} \quad \Omega_\phi = \int_0^R \frac{GM_\phi(r)}{r} dM(r), \quad (65)$$

where the quantity Ω is the usual gravitational potential energy of the system.

Finally, with the use of Eq. (62), we obtain the generalization of the virial theorem, in $f(R)$ modified theories of gravity, which takes the form

$$2K + \Omega - \Omega_\phi = 0. \quad (66)$$

Note that the generalized virial theorem, given by Eq. (66), can be written in an alternative form if we introduce the radii R_V and R_ϕ defined by

$$R_V = M^2 / \int_0^R \frac{M(r)}{r} dM(r), \quad \text{and} \quad R_\phi = M_\phi^2 / \int_0^R \frac{M_\phi(r)}{r} dM(r), \quad (67)$$

respectively. We denote R_ϕ as the *geometric radius* of the cluster of galaxies. Thus, the quantities Ω and Ω_ϕ are finally given by $\Omega = -GM^2/R_V$ and $\Omega_\phi = GM_\phi^2/R_\phi$, respectively.

The virial mass M_V is defined as

$$2K = \frac{GMM_V}{R_V}. \quad (68)$$

After substitution into the virial theorem, given by Eq. (66), we obtain

$$\frac{M_V}{M} = 1 + \frac{M_\phi^2 R_V}{M^2 R_\phi}. \quad (69)$$

If $M_V/M > 3$, a condition which is valid for most of the observed galactic clusters, then Eq. (69) provides the virial mass in $f(R)$ gravity, which can be approximated by

$$M_V \approx \frac{M_\phi^2 R_V}{M R_\phi}. \quad (70)$$

B. Geometric mass and geometric radius from galactic cluster observations

An interesting application of the generalized virial theorem can be inferred from the galaxy cluster observations. According to the modified $f(R)$ gravity model, the total mass of the cluster consists of the sum of the baryonic mass (mainly the intra-cluster gas), and the geometric mass, so that $M_{tot}(r) = 4\pi \int_0^r (\rho_g + \rho_\phi) r'^2 dr'$. Hence it follows that $M_{tot}(r)$ satisfies the following mass continuity equation

$$\frac{dM_{tot}(r)}{dr} = 4\pi r^2 \rho_g(r) + 4\pi r^2 \rho_\phi(r). \quad (71)$$

Note that most of the baryonic mass in the clusters of galaxies is in the form of the intra-cluster gas. The gas mass density ρ_g distribution can be fitted with the observational data by using the following expression for the radial baryonic mass (gas) distribution [70]

$$\rho_g(r) = \rho_0 \left(1 + \frac{r^2}{r_c^2}\right)^{-3\beta/2}, \quad (72)$$

where r_c is the core radius, and ρ_0 and β are (cluster-dependent) constants. Using the Jeans equation [28], one may obtain the total mass distribution [70, 71], so that taking into account the density profile of the gas given by Eq. (72), the total mass profile inside the cluster is given by

$$M_{tot}(r) = \frac{3k_B\beta T_g}{\mu m_p G} \frac{r^3}{r_c^2 + r^2}, \quad (73)$$

where k_B is Boltzmann's constant, T_g is the gas temperature, $\mu \approx 0.61$ is the mean atomic weight of the particles in the cluster gas, and m_p is the proton mass [70].

Using the mass continuity equation, Eq. (71), then Eqs. (72) and (73) provide the expression of the geometric density term inside the cluster, given by

$$4\pi\rho_\phi(r) = \frac{3k_B\beta T_g (r^2 + 3r_c^2)}{\mu m_p (r_c^2 + r^2)^2} - \frac{4\pi G\rho_0}{(1 + r^2/r_c^2)^{3\beta/2}}. \quad (74)$$

In the limit $r \gg r_c$ we obtain for ρ_ϕ the simple relation

$$4\pi\rho_\phi(r) = \left[\frac{3k_B\beta T_g}{\mu m_p} - 4\pi G\rho_0 r_c^{3\beta} r^{2-3\beta} \right] \frac{1}{r^2}, \quad (75)$$

and the geometric mass in the limit $r \gg r_c$, may be approximated as

$$GM_\phi(r) \approx \left[\frac{3k_B\beta T_g}{\mu m_p} - \frac{4\pi G\rho_0 r_c^{3\beta} r^{2-3\beta}}{3(1-\beta)} \right] r. \quad (76)$$

One may assume that the contribution of the gas density and mass to the geometric density and geometric mass, respectively, can be neglected. The latter approximations are very well supported by astrophysical observations, which show that the gas represents only a small fraction of the total mass [70, 72]. Therefore, we obtain

$$4\pi G\rho_\phi(r) \approx \left(\frac{3k_B\beta T_g}{\mu m_p} \right) r^{-2}, \quad \text{and} \quad GM_\phi(r) \approx \left(\frac{3k_B\beta T_g}{\mu m_p} \right) r, \quad (77)$$

respectively.

One may also estimate an upper bound for the cutoff of the geometric mass. The idea is to consider the point at which the decaying density profile of the geometric density associated to the galaxy cluster becomes smaller than the average energy density of the Universe. Let the value of the coordinate radius at the point where the two densities are equal to be $R_\phi^{(cr)}$. Then at this point $\rho_\phi(R_\phi^{(cr)}) = \rho_{univ}$, where ρ_{univ} is the mean energy density of the universe. By assuming $\rho_{univ} = \rho_c = 3H^2/8\pi G = 4.6975 \times 10^{-30} h_{50}^2 \text{ g/cm}^{-3}$, where $H = 50h_{50} \text{ km/Mpc/s}$ [70], we obtain

$$R_\phi^{(cr)} = \left(\frac{3k_B\beta T_g}{\mu m_p G \rho_c} \right)^{1/2} = 91.33 \sqrt{\beta} \left(\frac{k_B T_g}{5 \text{ keV}} \right)^{1/2} h_{50}^{-1} \text{ Mpc}. \quad (78)$$

The total geometric mass corresponding to this value is

$$M_\phi^{(cr)} = M_\phi(R_\phi^{(cr)}) = 4.83 \times 10^{16} \beta^{3/2} \left(\frac{k_B T_g}{5 \text{ keV}} \right)^{3/2} h_{50}^{-1} M_\odot. \quad (79)$$

This value of the mass is consistent with the observations of the mass distribution in the clusters of galaxies. However, according to $f(R)$ modified theories of gravity, we predict that the geometric mass and its effects extends beyond the virial radius of the clusters, which is of the order of only a few Mpc.

By assuming that $R_\phi \approx R_\phi^{(cr)}$, we obtain the following relation between the virial and the baryonic mass of the cluster

$$M_V \approx 91.33 \sqrt{\beta} \left(\frac{k_B T_g}{5 \text{ keV}} \right)^{1/2} h_{50}^{-1} \frac{M}{R_V (\text{Mpc})}. \quad (80)$$

For a cluster with gas temperature $T_g = 5 \times 10^7 \text{ K}$, $\beta = 1/2$ and $R_V = 2 \text{ Mpc}$ we obtain $M_V \approx 32M$, a relation which is consistent with the astronomical observations [70].

IV. SUMMARY AND DISCUSSION

Cosmology has entered a ‘golden age’, in which the rapid development of increasingly high-precision data has turned it from a speculative to an observationally based science. Recent experiments call upon state of the art technology to provide detailed information about the contents and history of the Universe. These experiments include the Hubble Space Telescope, the NASA WMAP satellite instrument, that measures the temperature and polarisation of the CMBR, and the Sloan Digital Sky Survey (SDSS), that is automatically mapping the properties and distribution of 1 million galaxies. High-precision cosmology has allowed us to tie down the parameters that describe our Universe with growing accuracy.

The standard model of cosmology is remarkably successful in accounting for the observed features of the Universe. However, there remain a number of fundamental open questions at the foundation of the standard model. In particular, we lack a fundamental understanding of the acceleration of the late universe. Recent observations of supernovae, together with the WMAP and SDSS data, lead to the remarkable conclusion that our universe is not just expanding, but has begun to accelerate [1]. What is the so-called ‘dark energy’ that is driving the acceleration of the universe? Is it a vacuum energy or a dynamical field (“quintessence”)? Or is the acceleration due to infra-red modifications of Einstein’s theory of General Relativity? How is structure formation affected in these alternative scenarios? What will the outcome be of this acceleration for the future fate of the universe?

The aspects of these fundamental questions whose resolution is so important for theoretical cosmology, need to look beyond the standard theory of gravity. It is clear that these questions involve not only gravity, but also particle physics. String theory provides a synthesis of these two parts of physics and is widely believed to be moving towards a viable quantum gravity theory. One of the key predictions of string theory is the existence of extra spatial dimensions. In the brane-world scenario, motivated by recent developments in string theory, the observed 3-dimensional universe is embedded in a higher-dimensional spacetime [29]. The generalization of the Dvali-Gabadadze-Porrati (DGP) brane models [31] lead to late-accelerating cosmologies [32], even in the absence of a dark energy field.

This exciting feature of “self acceleration” may help towards a new resolution to the dark energy problem, although this model deserves further investigation as a viable cosmological model [73]. It will be interesting to generalize the DGP model with the inclusion of a Gauss-Bonnet (GB) [33], and it will also be important to investigate the effects of the GB term relatively to the issues of strong coupling and ghosts in the DGP models. Infra-red modifications to General Relativity, where the consistency of various candidate models, including 4-dimensional modifications to the Einstein-Hilbert action, especially GB modifications with a scalar field coupling, have also been analyzed [57]. In this context, a more general modification of the Einstein-Hilbert gravitational Lagrangian density in the form of $L = f(R)$ has recently been extensively analyzed.

Relatively to the construction of “quintessential” dark energy models, recent fits to observational data indicate that an evolving equation of state crossing the phantom divide is mildly favored [74, 75]. In a cosmological setting, it has also been shown that the transition into the phantom regime, a mixture of various interacting non-ideal fluids is necessary [6], with important implications to the model construction of dark energy. If confirmed in the future, this behaviour holds important implications to the model construction of dark energy. The latter models, considering a redshift dependent equation of state, possibly provide better fits to the most recent and reliable SN Ia supernovae Gold dataset.

Deciding between these possible sources of the cosmic acceleration will be one of the major objectives in cosmology in the next decade with several surveys and experiments to address the nature of dark energy. One may mention new several major SNIa supernovae projects, such as the SuperNova Legacy Survey (SNLS), SDSS-II, Destiny, the Large Synoptic Survey Telescope (LSST), Dark Energy Survey (DES) and the SuperNova Acceleration Probe (SNAP). Other dark energy probes include the Dark UNiverse Explorer (DUNE); the Wide Field Multi-Object Spectrograph (WFMOs), which will perform surveys to measure dark energy and the history of our Galaxy; and the Panoramic Survey Telescope and Rapid Response System (PanSTARRS), amongst others. All of these aspects present an extremely fascinating aspect for the above-mentioned experiments and for future theoretical research.

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